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THE DRIFT OF A 24-HOUR EQUATORIAL SATELLITE DUE TO AN EARTH GRAVITY FIELD THROUGH 4th ORDER

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SUMMARY

This report extends previous investigations of 24-hour near equatorial earth satellites by considering the motion of such satellites in an earth gravity field through the 4th order. The three coupled second order linear differential equations of initial drift from a 24-hour equatorial circular reference orbit are presented. This linear system is analyzed for "stable regions" in the field within which drift motion is self limiting. A somewhat simplified form of the linear equations is integrated to give a general solution which defines the mean daily motion after injection to a high degree of accuracy.

CONTENTS

THE DRIFT OF A 24-HOUR EQUATORIAL SATELLITE DUE TO AN EARTH GRAVITY FIELD THROUGH 4th ORDER

(Manuscript received July 8, 1963)

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INTRODUCTION

Much analytic work has been done recently on the motions of a near synchronous satellite of the earth with a triaxial (2nd order) gravity field (References 1 and 2). The oscillatory movement of such a satellite about the minor axis of the earth's elliptical equator has been sufficiently well described. This investigation into higher order gravity effects on 24-hour satellites was prompted by a recent refinement of knowledge about these higher order anomalies (Reference 3). The perturbation forces arising from the higher order tesseral harmonics of the earth's gravity field are small, but they are in resonance on a 24-hour satellite. The major conclusion of the investigation is that, while no absolutely stationary geographic points exist for 24-hour satellites in an earth field to 4th order, the overall features of the regime of motion in the dominant triaxial field (see Reference 1 and pages 20-21) still hold.

DERIVATION OF THE INTEGRALS OF PERTURBATIONAL MOTION FOR A 24-HOUR EARTH SATELLITE

The force field F of the earth on a mass point $\underset{\cdot}{m}$ at earth centered r, ϕ, θ in inertial space (Figure 1) can be written:

$$\mathbf{F} = \underset{\cdot}{m} \left\{ \hat{r} \frac{\partial V_E}{\partial r} + \frac{\hat{\phi}}{r} \frac{\partial V_E}{\partial \phi} + \frac{\hat{\theta}}{r \cos\phi} \frac{\partial V_E}{\partial \theta} \right\}, \tag{1}$$

where

$$V_E = \frac{\mu_E}{r} \sum_{n=2}^{4} \sum_{m=0}^{n} \left[1 - \left(\frac{R_0}{r}\right)^n P_n^m (\sin\phi)\, J_{nm} \cos m\left(\theta - \theta_{nm}\right) \right] \tag{2}$$

(see Appendixes A and B). The XY plane is the earth's equatorial plane; θ_{nm} is the inertial longitude of the principal axis of symmetry of the earth's mass distribution accounted for by the nm harmonic of the geopotential V_E.

The acceleration of $\underset{\cdot}{m}$ in inertial space. r, ϕ, θ is

$$\vec{a} = \hat{r}\left(\ddot{r} - r\dot{\theta}^2 \cos^2\phi - r\dot{\phi}^2\right) + \hat{\phi}\left[\left(\frac{1}{r}\right)\left(\frac{d}{dt}\left[r^2\dot{\phi}\right]\right) + r\dot{\theta}^2 \cos\phi \sin\phi\right]$$

$$+ \hat{\theta}\left[\frac{1}{r\cos\phi}\frac{d}{dt}\left(r^2\dot{\theta}\cos^2\phi\right)\right]. \tag{3}$$

Writing F = ma, implies the following three scalar equations in the r, ϕ, θ components of force and acceleration:

$$\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2 = G_r\left(r, \phi, \theta - \theta_{nm}\right), \tag{4}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) + r\dot{\theta}^2\cos\phi\sin\phi = G_\phi\left(r, \phi, \theta - \theta_{nm}\right), \tag{5}$$

$$\frac{1}{r\cos\phi}\frac{d}{dt}\left(r^2\dot{\theta}\cos^2\phi\right) = G_\theta\left(r, \phi, \theta - \theta_{nm}\right), \tag{6}$$

where the G's are gravitational force components per unit mass (Appendix B).

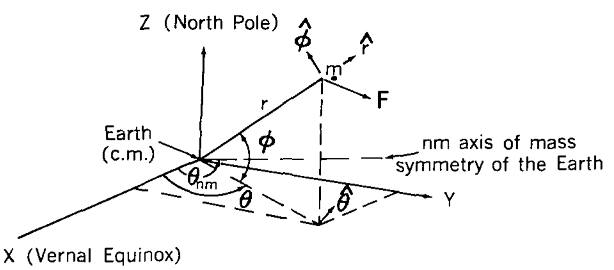

Figure 1—Coordinate system referencing the motion of a 24-hour earth satellite.

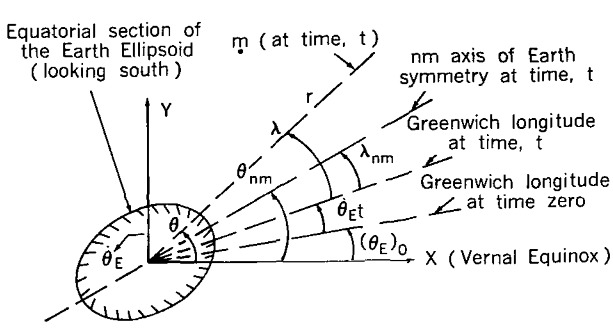

Figure 2—Section of the earth ellipsoid's equator showing the relationship of the various longitude references.

Consider the XY equatorial plane of the earth, with the earth's equator reflecting the mass distribution due to the nm harmonic of the geopotential (Figure 2). λ is the geographic longitude of $\underset{\cdot}{m}$; λ_{nm} is the geographic longitude of the principal nm axis of earth symmetry. Thus, it is clear from Figure 2 that $\lambda - \lambda_{nm} = \theta - \theta_{nm}$. The potential in Appendix B is thus consistent with that in Reference 4.

The reference orbit for the synchronous satellite is a circle in the equatorial plane of radius r_s, traversed at the earth's rotation rate. Therefore, we assume a perturbation solution to Equations 4, 5 and 6 of the following form:

$$\theta = [\text{Initial Inertial Longitude}] + [(\text{Earth Rate})\cdot(\text{Time})]$$

$$+ [\text{Geographic Longitude Perturbation}]$$

$$= \left[(\theta_E)_0 + \lambda_0\right] + \left[\dot{\theta}_E t\right] + [\Delta\lambda] ; \tag{7}$$

$$\phi = \Delta\phi ; \tag{8}$$

$$r = r_s + \Delta r , \tag{9}$$

where $\dot{\theta}_E$ is the time constant earth rotation rate, λ_0 is a constant equal to the initial geographic longitude position of $\dot{m}$, and $\Delta\lambda$ is the change in geographic longitude with time. Small r_s is a time-constant initial radius (the "synchronous radius") of $\dot{m}$, to be determined later, Δr the change in radius of $\dot{m}$ with time, and $(\theta_E)_0$ the initial inertial longitude of Greenwich. The perturbation (Equation 7) replaces $\theta - \theta_{nm}$ in the force field of Appendix B, by $(\theta_E)_0 + \lambda_0 + \dot{\theta}_E t + \Delta\lambda - \theta_{nm}$. But $\theta_{nm} = (\theta_E)_0 + \dot{\theta}_E t + \lambda_{nm}$, from Figure 2. Therefore, for the perturbation solutions, the force field longitude arguments are

$$\lambda_0 - \lambda_{nm} + \Delta\lambda .$$

Let

$$\lambda_0 - \lambda_{nm} = \gamma_{nm} \tag{10}$$

define harmonic constants referred to the initial geographic longitude of $\dot{m}$. Thus Equations 7, 8 and 9 are solutions to Equations 4, 5 and 6, if

$$\Delta\ddot{r} - (r_s + \Delta r)(\dot{\theta}_E + \Delta\dot{\lambda})^2 \cos^2\phi - (r_s + \Delta r)(\Delta\dot{\phi})^2 = G_r(r_s + \Delta r, \Delta\phi, \Delta\lambda + \gamma_{nm}), \tag{11}$$

$$\frac{1}{(r_s + \Delta r)} \frac{d}{dt} (r_s + \Delta r)^2 (\Delta\dot{\phi}) + (r_s + \Delta r)(\dot{\theta}_E + \Delta\dot{\lambda})^2 \cos\Delta\phi \sin\Delta\phi$$

$$= G_\phi (r_s + \Delta r, \Delta\phi, \Delta\lambda + \gamma_{nm}), \tag{12}$$

$$\frac{1}{(r_s + \Delta r)\cos\phi} \frac{d}{dt} (r_s + \Delta r)^2 (\dot{\theta}_E + \Delta\dot{\lambda}) \cos^2\Delta\phi = G_\theta (r_s + \Delta r, \Delta\phi, \Delta\lambda + \gamma_{nm}) . \tag{13}$$

Let $w = \dot{\theta}_E$, and by performing the indicated differentiations and substitutions, and ignoring all products of perturbations and/or perturbation rates, Equations 11-13 become

$$\ddot{r}_1 + ar_1 + b\dot{\lambda}_1 + c\lambda_1 + d\phi_1 = e \tag{14}$$

$$\ddot{\phi}_1 + f\phi_1 + gr_1 + h\lambda_1 = i \tag{15}$$

$$\ddot{\lambda}_1 + j\lambda_1 + k\dot{r}_1 + \ell r_1 + m\phi_1 = n, \tag{16}$$

where $r_1 = \Delta r/r_s$, $\lambda_1 = \Delta\lambda$, $\phi_1 = \Delta\phi$ are all dimensionless variables. The dots in Equations 14-16 and what follows refer to derivatives with respect to a dimensionless time $T = tw$. Thus

$$\frac{d}{dT} = \frac{1}{w}\frac{d}{dt}; \qquad \frac{d^2}{dT^2} = \frac{1}{w^2}\frac{d^2}{dt^2}.$$

The constants in Equations 14-16 are:

$$a = -1 - \frac{\mu_E}{w^2 r_s^3}\left\{2 + 6J_{20}\left(\frac{R_0}{r_s}\right)^2 - 36J_{22}\left(\frac{R_0}{r_s}\right)^2 \cos 2\gamma_{22} + 30\left(\frac{R_0}{r_s}\right)^3 J_{31}\cos\gamma_{31}\right.$$
$$- 300J_{33}\left(\frac{R_0}{r_s}\right)^2 \cos 3\gamma_{33} - 11.25J_{40}\left(\frac{R_0}{r_s}\right)^4 + 225J_{42}\left(\frac{R_0}{r_s}\right)^4 \cos 2\gamma_{42}$$
$$\left. - 3150J_{44}\left(\frac{R_0}{r_s}\right)^4 \cos 4\gamma_{44}\right\}; \tag{17}$$

$$b = -2; \tag{18}$$

$$c = \frac{\mu_E}{w^2 r_s^3}\left\{18J_{22}\left(\frac{R_0}{r_s}\right)^2 \sin 2\gamma_{22} - 6J_{31}\left(\frac{R_0}{r_s}\right)^3 \sin\gamma_{31} + 180J_{33}\left(\frac{R_0}{r_s}\right)^3 \sin 3\gamma_{33}\right.$$
$$\left. - 75J_{42}\left(\frac{R_0}{r_s}\right)^4 \sin 2\gamma_{42} + 2100J_{44}\left(\frac{R_0}{r_s}\right)^4 \sin 4\gamma_{44}\right\}; \tag{19}$$

$$d = \frac{\mu_E}{w^2 r_s^3}\left\{6J_{30}\left(\frac{R_0}{r_s}\right)^3 - 60J_{32}\left(\frac{R_0}{r_s}\right)^3 \cos 2\gamma_{32} + \frac{75}{2} J_{41}\left(\frac{R_0}{r_s}\right)^4 \cos\gamma_{41}\right.$$
$$\left. - 525J_{43}\left(\frac{R_0}{r_s}\right)^4 \cos 3\gamma_{43}\right\}; \tag{20}$$

$$e = 1 + \frac{\mu_E}{w^2 r_s^3}\left\{-1 - \frac{3}{2}\ J_{20}\left(\frac{R_0}{r_s}\right)^2 + 9J_{22}\left(\frac{R_0}{r_s}\right)^2 \cos 2\gamma_{22} - 6J_{31}\left(\frac{R_0}{r_s}\right)^3 \cos\gamma_{31}\right.$$

$$+ 60J_{33}\left(\frac{R_0}{r_s}\right)^3 \cos 3\gamma_{33} + \frac{15}{8}\ J_{40}\left(\frac{R_0}{r_s}\right)^4 - \frac{75}{2} J_{42}\left(\frac{R_0}{r_s}\right)^4 \cos 2\gamma_{42}$$

$$\left. + 525J_{44}\left(\frac{R_0}{r_s}\right)^4 \cos 4\gamma_{44}\right\}; \tag{21}$$

$$f = 1 - \frac{\mu_E}{w^2 r_s^3}\left\{-3J_{20}\left(\frac{R_0}{r_s}\right)^2 + 6J_{22}\left(\frac{R_0}{r_s}\right)^2 \cos 2\gamma_{22} - \frac{32}{2} J_{31}\left(\frac{R_0}{r_s}\right)^3 \cos\gamma_{31}\right.$$

$$+ 45J_{33}\left(\frac{R_0}{r_s}\right)^3 \cos 3\gamma_{33} + \frac{15}{2} J_{40}\left(\frac{R_0}{r_s}\right)^4 - 120J_{42}\left(\frac{R_0}{r_s}\right)^4 \cos 2\gamma_{42}$$

$$\left. + 420J_{44}\left(\frac{R_0}{r_s}\right)^4 \cos 4\gamma_{44}\right\}; \tag{22}$$

$$g = -\frac{\mu_E}{w^2 r_s^3}\left\{-9J_{30}\left(\frac{R_0}{r_s}\right)^3 + 90J_{32}\left(\frac{R_0}{r_s}\right)^3 \cos 2\gamma_{32} - \frac{105}{2} J_{41}\left(\frac{R_0}{r_s}\right)^4 \cos\gamma_{41}\right.$$

$$\left. + 735J_{43}\left(\frac{R_0}{r_s}\right)^4 \cos 3\gamma_{43}\right\}; \tag{23}$$

$$h = -\frac{\mu_E}{w^2 r_s^3}\left\{30J_{32}\left(\frac{R_0}{r_s}\right)^3 \sin 2\gamma_{32} - \frac{15}{2} J_{41}\left(\frac{R_0}{r_s}\right)^4 \sin\gamma_{41}\right.$$

$$\left. + 315J_{43}\left(\frac{R_0}{r_s}\right)^4 \sin 3\gamma_{43}\right\}; \tag{24}$$

$$i = \frac{\mu_E}{w^2 r_s^3}\left\{\frac{15}{2} J_{30}\left(\frac{R_0}{r_s}\right)^3 - 15J_{32}\left(\frac{R_0}{r_s}\right)^3 \cos 2\gamma_{32} + \frac{15}{2} J_{41}\left(\frac{R_0}{r_s}\right)^4 \cos\gamma_{41}\right.$$

$$\left. - 105J_{43}\left(\frac{R_0}{r_s}\right)^4 \cos 3\gamma_{43}\right\}; \tag{25}$$

$$j = -\frac{\mu_E}{w^2 r_s^3}\left\{12J_{22}\left(\frac{R_0}{r_s}\right)^2 \cos 2\gamma_{22} - \frac{3}{2} J_{31}\left(\frac{R_0}{r_s}\right)^3 \cos \gamma_{31} + 135J_{33}\left(\frac{R_0}{r_s}\right)^3 \cos 3\gamma_{33}\right.$$

$$\left. - 30J_{42}\left(\frac{R_0}{r_s}\right)^4 \cos 2\gamma_{42} + 1680J_{44}\left(\frac{R_0}{r_s}\right)^4 \cos 4\gamma_{44}\right\}; \tag{26}$$

$$k = 2. \tag{27}$$

$$l = -\frac{\mu_E}{w^2 r_s^3}\left\{- 30J_{22}\left(\frac{R_0}{r_s}\right)^2 \sin 2\gamma_{22} + 9J_{31}\left(\frac{R_0}{r_s}\right)^3 \sin \gamma_{31} - 270J_{33}\left(\frac{R_0}{r_s}\right)^3 \sin 3\gamma_{33}\right.$$

$$\left. + 105J_{42}\left(\frac{R_0}{r_s}\right)^4 \sin 2\gamma_{42} - 2940J_{44}\left(\frac{R_0}{r_s}\right)^4 \sin 4\gamma_{44}\right\}; \tag{28}$$

$$m = -\frac{\mu_E}{w^2 r_s^3}\left\{+ 30J_{32}\left(\frac{R_0}{r_s}\right)^3 \sin 2\gamma_{32} - \frac{15}{2} J_{41}\left(\frac{R_0}{r_s}\right)^4 \sin \gamma_{41}\right.$$

$$\left. + 315J_{43}\left(\frac{R_0}{r_s}\right)^4 \sin 3\gamma_{43}\right\}; \tag{29}$$

$$n = \frac{\mu_E}{w^2 r_s^3}\left\{6J_{22}\left(\frac{R_0}{r_s}\right)^2 \sin 2\gamma_{22} - \frac{15}{2} J_{31}\left(\frac{R_0}{r_s}\right)^3 \sin \gamma_{31} + 45J_{33}\left(\frac{R_0}{r_s}\right)^3 \sin 3\gamma_{33}\right.$$

$$\left. - 15J_{42}\left(\frac{R_0}{r_s}\right)^4 \sin 2\gamma_{42} + 420J_{44}\left(\frac{R_0}{r_s}\right)^4 \sin 4\gamma_{44}\right\}. \tag{30}$$

Writing Equations 14-16 in operator notation ($s^1 = d/dT$, $s^2 = d^2/dT^2$, $s = s^1$, $s^0 = 1$, etc.), we have

$$(s^2 + a)\, r_1 + (bs + c)\, \lambda_1 + (d)\, \phi_1 = e\ , \tag{31}$$

$$(g)\, r_1 + (h)\, \lambda_1 + (s^2 + f)\, \phi_1 = i\ , \tag{32}$$

$$(ks + l)\, r_1 + (s^2 + j)\, \lambda_1 + (m)\, \phi_1 = n\ . \tag{33}$$

Solving Equations 31-33 by Cramer's rule, we obtain

$$r_1 = \frac{\begin{vmatrix} e & bs + c & d \\ i & h & s^2 + f \\ n & s^2 + j & m \end{vmatrix}}{\begin{vmatrix} s^2 + a & bs + c & d \\ g & h & s^2 + f \\ ks + l & s^2 + j & m \end{vmatrix}}$$

$$= \left\{ e\left[hm - s^4 - jf - s^2(f + j)\right] - i\left[m(bs + c)\right.\right.$$

$$\left.\left. - d(s^2 + j)\right] + n(bs^3 + cs^2 + bfs + cf - h)\right\} \Big/ \left\{ \left[s^2 + a\right]\left[mh - s^4\right.\right.$$

$$\left. - s^2(j + f) - jf\right] - g\left[m(bs + c) - d(s^2 + j)\right] + (ks + l)\left(bs^3 + cs^2\right.$$

$$\left.\left. + bfs + cf - hd\right)\right\}. \tag{34}$$

Or

$$- s^6 r_1 + (A) s^4 r_1 + (B) s^3 r_1 + (C) s^2 r_1 + (D) s^1 r_1 + (E) s^0 r_1 = ehm + ejf - imc$$

$$+ idj + ncf - nhd. \tag{35}$$

This result follows from the evaluation of the determinants because $s(a, b, c, \cdots) = 0$, since a, b, c · · · · are all constants. In summation, the three uncoupled linear drift equations of sixth order are:

$$\begin{aligned} \left[- s^6 + (A) s^4 + (B) s^3 + (C) s^2 + (D) s^1 + (E) s^0\right] r_1 &= A_1 , \\ \left[- s^6 + (A) s^4 + (B) s^3 + (C) s^2 + (D) s^1 + (E) s^0\right] \lambda_1 &= A_2 , \\ \left[- s^6 + (A) s^4 + (B) s^3 + (C) s^2 + (D) s^1 + (E) s^0\right] \phi_1 &= A_3 . \end{aligned} \tag{36}$$

The constants are;

$$A = - a + kb - j - f , \tag{37}$$

$$B = kc + lb , \tag{38}$$

$$C = mh - aj - af - jf + gd + lc + bfk , \tag{39}$$

$$D = - gmb + kcf + lbf - khd , \tag{40}$$

$$E = amh - ajf - gmc + dgj + lcf - lhd , \tag{41}$$

$$A_1 = ehm + ejf - imc + idj + ncf - nhd , \tag{42}$$

$$A_2 = aim - anf - gem + gnd + lef - lid , \tag{43}$$

$$A_3 = ahn - aij - gcn + gej + lci - leh . \tag{44}$$

Examination of Equations 31-33 shows that for zero initial conditions $\left(r_1 = \dot{r}_1 = \phi_1 = \dot{\phi}_1 = \lambda_1 = \dot{\lambda}_1 = 0, \text{ at } T = 0\right)$; $\ddot{r}_1\ (T = 0) = e, \ddot{\lambda}_1\ (T = 0) = n$ and $\dot{\phi}_1\ (T = 0) = i$. The necessary and sufficient conditions for the drift to be zero for all time are, then, for the mass $\dot{m}$ to be placed with zero initial conditions into an orbit for which

$$e = n = i = 0 . \tag{45}$$

That this is so may be shown by successive differentiation of Equation 31-33 for the higher derivatives. They will all be zero providing only that the initial perturbation, perturbation rates and perturbation accelerations are zero. But Equations 45 are three transcendental equations in the two unknowns λ_0 and r_s (the initial longitude and radius of the satellite). Therefore, there will be, in general, no simultaneous solution except by coincidence of the constants of those equations. However, from what is known at present about the earth's gravitational field* (see Appendix C), the perturbation forces due to the latitude antisymmetry of the field (included in the i constant) are small compared to those in the radial and longitudinal directions at near synchronous altitudes. The latitude perturbations, then, may be neglected in considering the conditions for a near zero solution to Equations 31-33. A plot of these perturbation forces with λ_0 at an r_s determined from $e = 0$, is found in Appendix C. It is postulated then, and proved later, that essentially stable regions of the gravity field in geographic-geocentric coordinates exist in the neighborhood of one or more points on the equator for which

$$e = n = 0 . \tag{46}$$

*Kozai, Y., Private Communication, November 1963.

In the development which follows, the earth gravity field of Kozai is used together with an earth rotation rate from Reference 3. These earth constants are:*

$$
\left.\begin{aligned}
w &= .7292115 \times 10^{-4} \text{ rad./sec.} \qquad \text{(Reference 3)} \\
\mu_E &= 3.986032 \times 10^{20} \text{ cm.}^3\text{/sec.}^2 \\
R_0 &= 6.378165 \times 10^{8} \text{ cm.} \\
J_{20} &= 1082.48 \times 10^{-6} \\
J_{30} &= -2.56 \times 10^{-6} \\
J_{40} &= -1.84 \times 10^{-6} \\
J_{21} &= 0. \\
J_{22} &= -1.2 \times 10^{-6} \\
J_{31} &= -1.9 \times 10^{-6} \\
J_{32} &= -.14 \times 10^{-6} \\
J_{33} &= -.10 \times 10^{-6} \\
J_{41} &= -.52 \times 10^{-6} \\
J_{42} &= -.062 \times 10^{-6} \\
J_{43} &= -.035 \times 10^{-6} \\
J_{44} &= -.031 \times 10^{-6} \\
\lambda_{22} &= -26.4 \text{ degrees} \\
\lambda_{31} &= 4.6 \text{ degrees} \\
\lambda_{32} &= -16.8 \text{ degrees} \\
\lambda_{33} &= 42.6 \text{ degrees} \\
\lambda_{41} &= 237.5 \text{ degrees} \\
\lambda_{42} &= 65.2 \text{ degrees} \\
\lambda_{43} &= 0.5 \text{ degrees} \\
\lambda_{44} &= 14.9 \text{ degrees}
\end{aligned}\right\} \qquad (47)
$$

*The constants are taken from Kozai, Y., Private Communication, November 1962, unless otherwise noted.

The two transcendental equations in λ_0 and r_s arising from Equation 46 are:

$$
\begin{aligned}
\frac{w^2 r_s^3}{\mu_E} = {} & 1 + \frac{3}{2} J_{20}\left(\frac{R_0}{r_s}\right)^2 + \cos\lambda_0 \left\{- 6J_{31}\left(\frac{R_0}{r_s}\right)^3 \cos\lambda_{31}\right\} + \cos 2\lambda_0 \left\{- 9J_{22}\left(\frac{R_0}{r_s}\right)^2 \cos 2\lambda_{22}\right. \\
& \left. + \frac{75}{2} J_{42}\left(\frac{R_0}{r_s}\right)^4\right\} + \cos 3\lambda_0 \left\{- 60J_{33}\left(\frac{R_0}{r_s}\right)^3 \cos 3\lambda_{33}\right\} + \cos 4\lambda_0 \\
& \cdot \left\{- 525J_{44}\left(\frac{R_0}{r_s}\right)^4 \cos 4\lambda_{44}\right\} - \frac{15}{8} J_{40}\left(\frac{R_0}{r_s}\right)^4 + \sin\lambda_0 \left\{- 6J_{31}\left(\frac{R_0}{r_s}\right)^3 \sin\lambda_{31}\right\} \\
& + \sin 2\lambda_0 \left\{- 9J_{22}\left(\frac{R_0}{r_s}\right)^2 \sin 2\lambda_{22} + \frac{75}{2} J_{42}\left(\frac{R_0}{r_s}\right)^4 \sin 2\lambda_{42}\right\} \\
& + \sin 3\lambda_0 \left\{- 60J_{33}\left(\frac{R_0}{r_s}\right)^3 \sin 3\lambda_{33}\right\} + \sin 4\lambda_0 \left\{- 525J_{44}\left(\frac{R_0}{r_s}\right)^4 \sin 4\lambda_{44}\right\} ; \qquad (48)
\end{aligned}
$$

and

$$
\begin{aligned}
0 = {} & \sin\lambda_0 \left[- \frac{3}{2} J_{31}\left(\frac{R_0}{r_s}\right) \cos\lambda_{31}\right] + \sin 2\lambda_0 \left[6J_{22} \cos 2\lambda_{22} - 15J_{42}\left(\frac{R_0}{r_s}\right)^2\right] \\
& + \sin 3\lambda_0 \left[45J_{33}\left(\frac{R_0}{r_s}\right) \cos 3\lambda_{33}\right] + \sin 4\lambda_0 \left[420J_{44}\left(\frac{R_0}{r_s}\right)^2 \cos 4\lambda_{44}\right] \\
& + \cos\lambda_0 \left[\frac{3}{2} J_{31}\left(\frac{R_0}{r_s}\right) \sin\lambda_{31}\right] + \cos 2\lambda_0 \left[- 6J_{22} \sin 2\lambda_{22} + 15J_{42}\left(\frac{R_0}{r_s}\right)^2\right. \\
& \left. \sin 2\lambda_{42}\right] + \cos 3\lambda_0 \left[- 45J_{33}\left(\frac{R_0}{r_s}\right) \sin 3\lambda_{33}\right] + \cos 4\lambda_0 \left[- 420J_{44}\left(\frac{R_0}{r_s}\right)^2 \sin 4\lambda_{44}\right] . \qquad (49)
\end{aligned}
$$

In general, for every finite non zero r_s, Equation 49 changes sign a minimum of two times and a maximum of eight times over the equator. The earth constants are such that for $r_s > R_0$, the right hand side of Equation 48 is very close to 1 for all points on the equator. Essentially then, Equations 48 and 49 decouple. Equation 48 may be solved separately for a near synchronous radius independent of longitude, leaving a small longitude dependent residual. With the near synchronous radius so determined, the zero's of Equation 49 establish (to high accuracy) a minimum of two and a maximum of 8 potentially drift free points on the equator. The longitude dependent residual of Equation 48 may then be solved to establish (to high accuracy) the radius to each potentially drift free point. The perturbation constants of the earth's gravity field are so much less than 1 that the iteration need not be carried further than that outlined above (see Appendix D). The results of this iteration of Equation 48 and 49 with the earth constants of Equation 47 are: The spherical earth "synchronous radius" is

$$r_s \text{ (Spherical)} = 138333942.5 \text{ ft. (26199.6103 statute miles)}; \tag{50}$$

The "oblate earth" (including J_{20} and J_{40} potential terms) "synchronous radius" is

$$r_s \text{ (Oblate)} = 138335648.5 \text{ ft. (26199.9334 statute miles)}. \tag{51}$$

With the aforementioned oblate earth synchronous radius, Equation 49 becomes

$$0 = .430 \sin \lambda_0 - 4.37 \sin 2\lambda_0 + .4175 \sin 3\lambda_0 - .151 \sin 4\lambda_0 - .0345 \cos \lambda_0$$

$$- 5.76 \cos 2\lambda_0 + .538 \cos 3\lambda_0 + .257 \cos 4\lambda_0 . \tag{52}$$

The zero's of Equation 52, which are the potentially drift free longitudes around the equator, are at

$$\lambda_0 = 64.2°, \ 155.8°, \ 242.9° \text{ and } 331.3°. \tag{53}$$

It is interesting to compare these potentially stable longitudes with those which would be present if all the tesseral and sectorial harmonics except J_{22} are ignored. In the simpler field (the so-called "triaxial" gravity field), the zero's of Equation 48 with the oblate earth synchronous radius of Equation 51 are at

$$\lambda_0 = 63.6°, \ 153.6°, \ 243.6° \text{ and } 333.6°. \tag{54}$$

In no case (with a full earth potential) do the "stable" longitudes differ by more than 2.3° from those which exist in the simpler "triaxial" field.

The potentially drift free radii to the longitudes of Equation 53 are:

$$r_s(\lambda_0 = 64.2°) = 138335637.5 \text{ ft. (26199.9313 statute miles)} \tag{55a}$$

$$r_s(\lambda_0 = 155.8°) = 138335660.2 \text{ ft. (26199.9356 statute miles)} \tag{55b}$$

$$r_s(\lambda_0 = 242.9°) = 138335635.2 \text{ ft. (26199.9309 statute miles)} \tag{55c}$$

$$r_s(\lambda_0 = 331.3°) = 138335660.3 \text{ ft. (26199.9356 statute miles)}. \tag{55d}$$

The uncoupled linearized drift Equations 36 may be simplified and integrated directly by ignoring all terms in the differential coefficients A, B, C, D and E which are much less than 1. The driving terms

A_1, A_2, A_3 must retain at least one order of smallness less than 1 so that the resulting solution is sufficiently sensitive to drift acceleration. The initial radius may be chosen for convenience as the mean of those in Equation 55. But to insure the longest possible validity for the resulting solution, it is probably best to solve $e = 0$ for r_s at the λ_0 from which the perturbation is desired. In any case, $r_s = 138335647.7 \pm 12.5$ ft. for near zero solutions to Equation 36 with zero initial conditions. The simplified uncoupled drift equations then become:

$$\begin{aligned} \left(s^6 + 2s^4 + s^2\right) r_1 &= -A_1 \\ \left(s^6 + 2s^4 + s^2\right) \lambda_1 &= -A_2 \\ \left(s^6 + 2s^4 + s^2\right) \phi_1 &= -A_3 \,. \end{aligned} \tag{56}$$

It may be verified that the complete solution to Equation 56 is:

$$\Delta_q = C_{1q} + C_{2q}T + \left(C_{3q} + C_{4q}T\right) \sin T + \left(C_{5q} + C_{6q}T\right) \cos T - \frac{A_q T^2}{2} \;, \tag{57}$$

where

$$\begin{aligned} \Delta_q &= r_1 \text{ when } q = 1 \,, \\ \Delta_q &= \lambda_1 \text{ when } q = 2 \,, \\ \Delta_q &= \phi_1 \text{ when } q = 3 \,. \end{aligned}$$

In any dynamics problem utilizing the perturbation solutions (Equation 57), 18 conditions on the perturbations must be specified.

The coefficients C_{iq} of the approximate perturbation solution (Equation 57) for the synchronous equatorial satellite have been solved in Appendix A of Reference 5 for the general case where any small initial perturbations and perturbation rates may be given. Ignoring terms of second and lower order smallness, the linearized drift solution for the 24-hour equatorial satellite is approximately

$$\begin{aligned} r_1 = {} & 4s_0^0 r_1 + 2s_0^1 \lambda_1 + e + 2nT + \left[s_0^1 r_1 - 2n - \left(125_0^0 r_1\right) T\right] \sin T \\ & + \left[-3s_0^0 r_1 - 2s_0^1 \lambda_1 - e\right] \cos T \;; \end{aligned} \tag{57a}$$

$$\lambda_1 = -2s_0^1 r_1 + s_0^0\lambda_1 + 4n + \left[18s_0^0 r_1 - 3s_0^1\lambda_1 - 2e\right]T + \left(-24s_0^0 r_1 + 4s_0^1\lambda_1 + 2e\right)\sin T + \left[2s_0^1 r_1 - 4n + \left(6s_0^0 r_1\right)T\right]\cos T - \frac{3}{2}nT^2 \; ; \tag{57b}$$

$$\phi_1 = i + \left[s_0^1\phi_1\right]\sin T + \left[s_0^0\phi_1 - i\right]\cos T \tag{57c}$$

The $s_0^0\Delta_q$ are initial dimensionless perturbations; the $s_0^1\Delta_q$ are initial dimensionless perturbation rates.

DRIFT OF A 24-HOUR EQUATORIAL SATELLITE WITH LOW INITIAL RATES

The general character of the drift following near perfect injection at radii for which e is equal to or close to zero, may be found by examining the approximate integrals (Equations 57a, 57b, 57c). There is a coupled daily harmonic oscillation in the drift motion due to the eccentricity introduced principally by the resonant longitudinal perturbation force (when present). This daily harmonic oscillation is also partly due to coupling from the latitude and radial perturbations. The long term drift in radius is controlled by the $2nT$ term for perfect injection $\left(s_0^1 = s_0^0 = 0\right)$. The long term drift in longitude is controlled by the $-3n/2/T^2$ term for perfect injection. The sign of n changes four times around the equator. Table 1 shows the dominant drift effect following a near perfect "synchronous" injection. Thus the long term radial and longitudinal drift following near perfect injection at near synchronous radii $(e \approx 0)$ around the equator has the character (not to scale) shown in Figure 3. The long term motion in longitude and radius is thus highly suggestive of the coupled long term circulation about the minor axis of the earth ellipsoid's equator predicted* and confirmed in the computer studies in Reference 1; both studies made with a triaxial gravitational field. All the results of this study showing the dominance of the J_{22} (triaxial) term in the longitudinal perturbations, point strongly to the conclusion that

Table 1
The Dominant Drift Effect Following a Near Perfect "Synchronous" Injection

Longitude Range (degrees)	Sign (n) (dominant drift in radius)	Sign (−n) (dominant drift in longitude)
$62.4 < \lambda_o < 155.8$:	+	-
$155.8 < \lambda_o < 242.9$:	-	+
$242.9 < \lambda_o < 331.3$	+	-
$331.3 < \lambda_o < 62.4$:	-	+

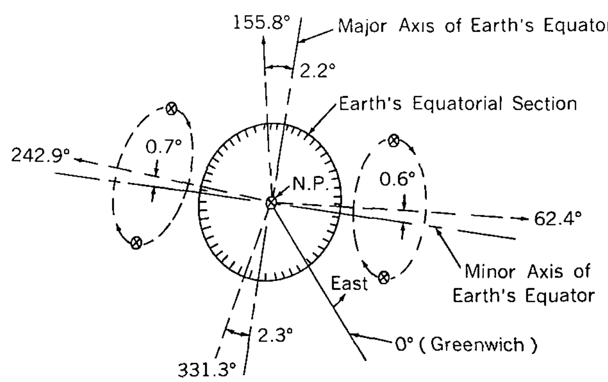

Figure 3—Long term radial and longitudinal drift following near perfect injection at near synchronous radii around the equator.

*Frick and Garber, Private Communication, 1962.

such a circulation about two dynamically "stable" regions of zero longitude and radial force perturbation, exists for the full earth potential as well.

The latitude drift following a near perfect "synchronous" injection, predicted in (57c) is initially a daily harmonic oscillation of small amplitude, but predominantly on one side of the equator or the other depending on the sign of i which is proportional to the latitude perturbing force at the equator. Since J_{30} (the so-called "pear shaped" harmonic of the earth's field) dominates i and it is negative, the perturbing force is directed south and the initial daily oscillation is southerly with an amplitude of $2i$.

The drift of a near synchronous earth satellite injected with zero drift rates at -71.4° longitude from Greenwich (45° from the minor axis of the earth ellipsoid's equator, and thus having close to maximum longitude perturbations) as predicted by Equations 57a, 57b, 57c, is

$$\Delta r = 259t - 41.25 \sin(2\pi t)\,\text{ft},(t, \text{in sidereal days}); \tag{57d}$$

$$\Delta\lambda = 34.2 \times 10^{-6}(1 - \cos 2\pi t) - 505 \times 10^{-6} t^2 \text{ degrees},(t, \text{ in sidereal days}); \tag{57e}$$

$$\Delta\phi = -10.7 \times 10^{-7}(1 - \cos 2\pi t) \text{ degrees},(t, \text{ in sidereal days}). \tag{57f}$$

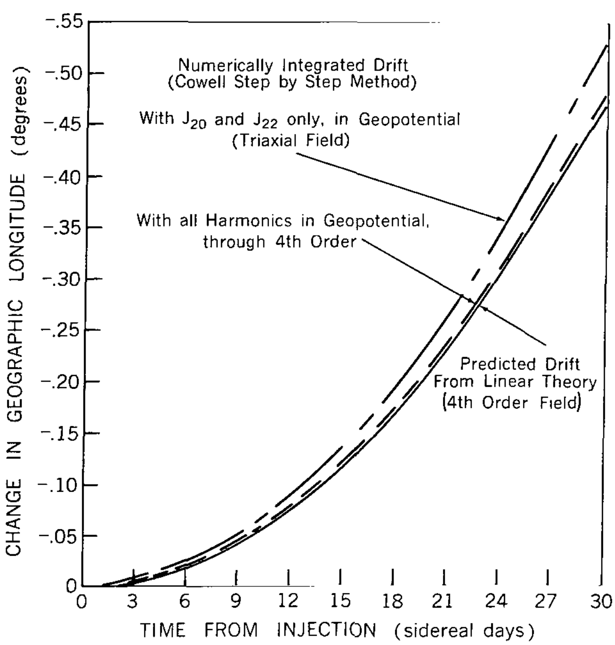

Figure 4—Comparison of numerically integrated and predicted longitude drift of a near 24-hour satellite injected at -71.4° (*45° east of the earth ellipsoid's minor* equatorial axis) at a radius of 138,335,650.2 feet, with zero initial perturbation rates.

Numerical integration (Cowell step by step method) of the equations of motion have been carried out on an IBM 7090 computer and the comparison with the predictions of Equations 57d, 57e, 57f is illustrated in Figures 4 and 5 for the longitudinal and radial drifts to 30 days following injection. The full results indicate that the linear theory of Equation 57 will predict the perturbations of a near synchronous, near equatorial satellite due to a "full" earth potential through 4th order, to within 2 percent in the mean daily longitude drift, and 1 percent in the mean daily radial drift for up to 180 days following a near perfect injection. The numerically computed latitude drift of the near synchronous satellite is of the order of $\pm 10^{-6}$ degrees, maximum, over 180 sidereal days, which agrees with (57f) in order of magnitude. Numerically integrated drifts for the above example in a "triaxial" (J_{20} and J_{22} harmonics only) earth field show errors of about 10 percent from the drift in a full potential field in both radius and longitude to 180 days following injection. This "error" was reduced to about 5 percent with the inclusion of the J_{31} harmonic in the programmed potential function. Inclusion of the J_{33} harmonic reduces the "error" to about 1 percent over 180 days.

It should be noted that the linear theory presented in this report does *not* predict the change in eccentricity with time of the spiralling orbit of the resonant near synchronous satellite. However, the results of the numerical integration on the above example show that the *initial* eccentricity of the spiral orbit is well predicted. The radius in the numerically integrated orbit has a daily oscillation of about $\pm(34 + 5t)$ feet, (t, in days) for a period of 5 days following injection; which is in excellent agreement with (74a) considering the inherent machine error in the numerical integration and the simplifications in the theory.

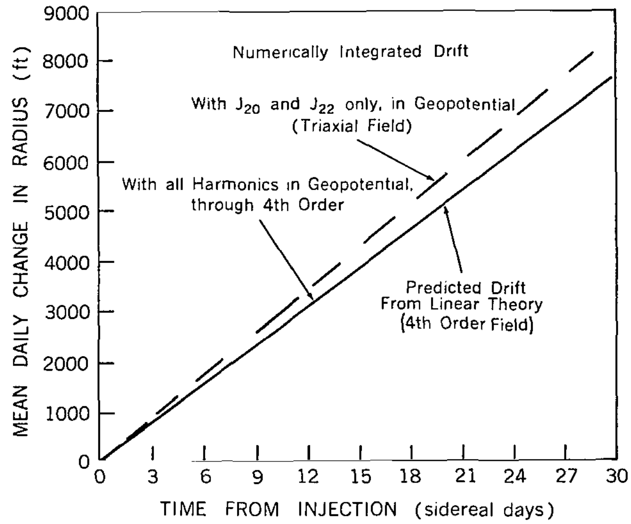

Figure 5—Comparison of numerically integrated and predicted mean daily radial drift of a near 24-hour satellite injected at -71.4° at a radius of 138,335,650.2 feet, with zero initial perturbation rates.

STABILITY OF MOTIONS OF THE 24-HOUR SATELLITE NEAR POINTS OF ZERO LONGITUDINAL AND RADIAL PERTURBATION FORCES

If the longitudinal and radial perturbation forces are zero, it may be verified that $e = n = 0$ (see Appendixes B and C) and the linearized drift Equations 31-33 become

$$\left.\begin{aligned} (s^2 + a)r_1 + (bs + c)\lambda_1 + (d)\,\phi_1 &= 0\ ; \\ (g)r_1 + (h)\lambda_1 + (s^2 + f)\,\phi_1 &= i\ ; \\ (ks + l)r_1 + (s^2 + j)\lambda_1 + (m)\,\phi_1 &= 0\ . \end{aligned}\right\} \tag{58}$$

The characteristic equation of the system (Equations 58) is (evaluating the differential constants of Equation 36 for $e = n = 0$)

$$q^6(-1) + q^4(-a + kb - j - f) + q^3(kc + lb) + q^2(mh - aj - af - jf + gd + lc + bfk)$$

$$+ q(-gmb + kcf + lbf - khd) + amh + dgj + lcf - lhd = 0\ . \tag{59}$$

The particular solutions to Equation 58 will be constants:

$$\left.\begin{aligned} r_1 \text{ (particular)} &= \frac{E}{A_1} ; \\ \lambda_1 \text{ (particular)} &= \frac{E}{A_2} ; \\ \phi_1 \text{ (particular)} &= \frac{E}{A_3} . \end{aligned}\right\} \qquad (60)$$

To simplify the calculation of the roots of the characteristic Equation 59 without losing anything essential in the characteristics of the motion, we will ignore the lower order perturbation constants and/or products of perturbation constants in each coefficient of Equation 59. With this simplification and with the large constants evaluated, Equation 59 becomes

$$q^6(-1) + q^4(-2) + q^3(2c - 2l) + q^2(-1) + q(2c - 2l) + 3j = 0 . \qquad (61)$$

It is noted that the characteristic values of the motion from Equation 61 are independent of the latitude perturbations of the earth's gravity field. Consider $q_{1,2}$ as solutions to Equation 61, where $|q_{1,2}| << 1$. For these solutions, ignore orders of q smaller than q^2. Equation 61 in $q_{1,2}$ then becomes;

$$q^2 + 2(l - c)q - 3j = 0 . \qquad (62)$$

The solutions to Equation 62 are

$$q_{1,2} = -(l - c) \pm \left[(l - c)^2 + 3j\right]^{1/2} . \qquad (63)$$

Since $|l - c|, |3j| << 1$, the approximation to Equation 62 for these roots is valid.

There are also solutions to Equation 61: $|q| \approx (-1)^{1/2}$. Therefore, let

$$q_{3,4} = +(-1)^{1/2} + \epsilon_{3,4} , \qquad (64)$$

where $|\epsilon_{3,4}| << 1$. With Equation 64 substituted into Equation 61, Equation 61 becomes

$$\left[(-1)^{1/2} + \epsilon_{3,4}\right]^6 + 2\left[(-1)^{1/2} + \epsilon_{3,4}\right]^4 + 2(l - c)\left[(-1)^{1/2} + \epsilon_{3,4}\right]^3$$

$$+ \left[(-1)^{1/2} + \epsilon_{3,4}\right]^2 + \left[(-1)^{1/2} + \epsilon_{3,4}\right]\left[2(l - c)\right] - 3j = 0 . \qquad (65)$$

Since $|\epsilon_{3,4}| << 1$, the expansion of Equation 65 ignoring terms in ϵ^3 and higher, is

$$6\epsilon\left[(-1)^{1/4}\right]^5 + 8\epsilon\left[(-1)^{1/4}\right]^3 + 6(l-c)\,\epsilon\left[(-1)^{1/4}\right]^2 + 2\epsilon(-1)^{1/4} - 2(l-c)(-1)^{1/4}$$

$$+ 2\epsilon(l-c)(-1)^{1/4} + 2(l-c)(-1)^{1/4} + 15\epsilon^2 - 12\epsilon^2 + 6(l-c)\,\epsilon^2(-1)^{1/4}$$

$$+ \epsilon^2 = 0 .$$

This last equation reduces to

$$\epsilon^2\left[15 - 12 + 6(l-c)(-1)^{1/4} + 1\right] + \epsilon\left[6(-1)^{1/4} - 8(-1)^{1/4} - 6(l-c) + 2(-1)^{1/4}\right.$$

$$\left. + 2(l-c)(-1)^{1/4}\right] = 0 .$$

Thus, the approximate roots to Equation 65 are

$$\epsilon_3 = 0 , \qquad \epsilon_4 = \frac{-2(l-c)\left[-3+(-1)^{1/4}\right]}{4} . \tag{66}$$

Similarly, letting

$$q_{5,6} = -(-1)^{1/4} + \epsilon_{5,6} . \tag{67}$$

The expansion of Equation 61, with the same order of approximation as for $\epsilon_{3,4}$, reduces to

$$6\epsilon\left[-(-1)^{1/4}\right]^5 + 8\epsilon\left[-(-1)^{1/4}\right]^3 + 6(l-c)\,\epsilon\left[-(-1)^{1/4}\right]^2 + 2\epsilon\left[-(-1)^{1/4}\right] + 2(l-c)(-1)^{1/4}$$

$$- 2\epsilon(l-c)(-1)^{1/4} - 2(l-c)(-1)^{1/4} + 15\epsilon^2 - 12\epsilon^2 - 6(l-c)\,\epsilon^2(-1)^{1/4} + \epsilon^2 = 0 .$$

The roots to the above equation are

$$\epsilon_5 = 0 , \qquad \epsilon_6 \doteq \frac{-2(l-c)\left[-(-1)^{1/4}-3\right]}{4} . \tag{68}$$

Thus Equations 66 and 68 substituted into Equations 64 and 67 determine the other four roots to Equation 61 as approximately

$$q_{3,5} = \pm(-1)^{1/4} , \qquad q_{4,6} = \pm(-1)^{1/4}\left[1 - \frac{1}{2}(l-c)\right] + \frac{3}{2}(l-c) . \tag{69}$$

Stability in the Triaxial Geopotential Field

In this case $l = c = 0$ at the points of zero longitudinal and radial perturbation forces. The four characteristic solutions are

$$q_{1,2}\ (\text{triaxial}) \doteq \pm (3j)^{1/2}\ , \tag{70a}$$

$$q_{3,4}\ (\text{triaxial}) \doteq \pm (-1)^{1/2}\ . \tag{70b}$$

While it is true that there are two more independent complementary solutions to Equation 58 that may be found for the triaxial case, their constants will be found to be zero when a natural set of initial conditions are specified. For the triaxial case, $g = h = d = i = m = 0$. Therefore Equation 58 uncouples in the latitude variation and there are only four characteristic solutions to the set in the longitude and radial perturbations. It may be shown that (70a) and (70b) are just these four. The redundant $q_{5,6} = \pm(-1)^{1/2}$ are the two characteristic solutions to the latitude variation in the triaxial case.

On or Near the Major Axis in the Triaxial Field

$$3j\left(\lambda_o = 153.6°,\ -26.4°\right) = +0.99 \times 10^{-6}$$

The motion thus has a slowly divergent component.

On or Near the Minor Axis in the Triaxial Field

$$3j\left(\lambda_o = 63.6°,\ -116.4°\right) = -0.99 \times 10^{-6}$$

The motion consists of two non-damped self-limiting oscillations. One has a period in the neighborhood of one day (from $q_{3,4}$). The other has a frequency of $(.99)^{1/2} \times 10^{-3} = 0.995 \times 10^{-3}$ (dimensionless) $= .995 \times 10^{-3}\,w$ (dimensions of time^{-1}). This long period oscillation has a period in the neighborhood of

$$\frac{2\pi(\text{rad./cycle})}{.995 \times 10^{-3} \times 2\pi(\text{rad./day})} = 1005 \text{ days} = 2.76 \text{ years}\ .$$

Stability in the Geopotential Field Through 4th Order

For motion in the vicinity of the 4 points of zero longitudinal and radial perturbation forces:

1. There will be 4 characteristic solutions giving damped oscillations with periods near one day $\left[q_{3,4,5,6}\text{ from Equation 69}\right]$

2. There will be 2 characteristic solutions giving either a weak-negatively damped oscillation with a period near 1000 days, or two exponentials, one of which is slowly divergent in character. $\left[q_{1,2}\right.$ from Equation 63$\left.\right]$

For Motion in the Vicinity of the Zero Perturbations at $\lambda_o = 64.2°$

$$l - c = -0.36 \times 10^{-9}, \qquad 3j = -1.026 \times 10^{-6}$$

Perfect injection in the vicinity of this quasi-stable point is thus followed by two self-limiting oscillations with periods in the vicinity of one day which are weak-positively damped $(q_{3,4,5,6})$. There is also a self-limiting oscillation with a long period of $1/(1.026)^{1/2} \times 10^{-3}$ = 987 Days = 2.71 Years, which is weak-negatively damped $(q_{1,2})$.

For Motion in the Vicinity of the Zero Perturbations at $\lambda_o = 155.8°$

$3j$ is controlled by the J_{22} term, is greater than zero, and $\pm(3j)^{1/2}$ is of the order of $\pm 10^{-3}$. $|l - c|$ is of the order of 10^{-9} (as before), so that one of the characteristic solutions will be a slowly divergent exponential.

For Motion in the Vicinity of the Zero Perturbations at $\lambda_o = 242.9°$ *(-117.1°)*

$$l - c = +5.3 \times 10^{-9}, \qquad 3j = -1.123 \times 10^{-6}.$$

Perfect injection in the vicinity of this quasi-stable point is thus followed by two self-limiting coupled oscillations with periods in the vicinity of one day which are weak-negatively damped $(q_{3,4,5,6})$. There is also a coupled self-limiting oscillation with a long period near $1/(1.123)^{1/2} \times 10^{-3}$ = 944 days = 2.58 year, which is weak-positively damped $(q_{1,2})$.

For Motion in the Vicinity of the Zero Perturbations at $\lambda_o = 331.3°$ *(-28.7°)*

$3j$, controlled by the J_{22} term, is greater than zero. $\pm(3j)^{1/2}$ is of the order of $\pm 10^{-3}$. $|l - c|$ is of the order of 10^{-9} as before. Therefore, the resultant motion is slowly divergent in character as one of the $q_{1,2}$ solutions will be positive of the order of $+10^{-3}$.

In conclusion, for the geopotential gravity field through 4th order, two regions on the equator have been found within which small initial perturbations in geographic longitude, latitude, and radius are self limiting in the sense that the resulting motion of an earth satellite in these regions is essentially harmonic in character with very weak damping. These regions are in the neighborhood of 64.2° East of Greenwich and 26,199.9314 statute miles and 117.1° West of Greenwich and 26,199.9309 statute miles from the center of mass of the earth. They are both within 2-1/2 degrees of the minor axis of the earth ellipsoid's equator. The damping is of the order of 10^{-9} ωt, or of the order of 10^{-8} t (days). Thus, initial amplitudes of the damped harmonic perturbations in these two regions suffer a twofold change in magnitude in the order of $t = \ln(2)/10^{-8} = 6.93 \times 10^{7}$ days $= 1.9 \times 10^{5}$ years. These regions then, can be considered to be essentially stable.

While the regions around 64.2° and 242.9° on the equator have an inherent dynamic stability in the sense shown above, the regions around the zero perturbation points at 155.8° and 331.3° are only very weakly dynamically unstable. At the $n = e = 0$ point where $\lambda_o = 331.3°$ and $r_s = 138{,}335{,}660.3$ ft., by assuming zero initial perturbations and rates, the resultant motion from a complete evaluation of the C_{iq} coefficients of Equation 57 in Appendix A of Reference 5, is

$$r_1 \doteq -1.63 \times 10^{-16} + (-1.76 \times 10^{-16} T) + (+2.64 \times 10^{-16} - .815 \times 10^{-16} T) \sin T$$

$$+ (1.63 \times 10^{-16} - .88 \times 10^{-16} T) \cos T - 21.3 \times 10^{-24} T^2 , \qquad (71a)$$

$$\lambda_1 \doteq -6.15 \times 10^{-16} + (3.26 \times 10^{-16} T) + (-4.89 \times 10^{-16} + 1.76 \times 10^{-16} T) \sin T$$

$$+ (+6.15 \times 10^{-16} + 1.63 \times 10^{-16} T) \cos T + 13.2 \times 10^{-17} T^2 , \qquad (71b)$$

$$\phi_1 \doteq -6.4 \times 10^{-9} + (-23.7 \times 10^{-14} T) \sin T + (6.4 \times 10^{-9} \cos T) + 24.5 \times 10^{-16} T^2 . \qquad (71c)$$

Equation 71a predicts a change in injection radius of only -.0003 feet in 5 years. Equation 71b predicts a change in injection geographic longitude of only $+1.00 \times 10^{-6}$ degrees in 5 years. Equation 71c predicts a change in injection latitude of only $+1.85 \times 10^{-5}$ degrees in 5 years. These are all mean daily drifts. The amplitudes of the daily oscillations are even smaller quantities. The conclusion is that station keeping requirements for near synchronous satellites placed with low initial rates near the major axis of the earth ellipsoid's equator, will be virtually unaffected by the non-central character of the earth's gravitational field.

CONCLUSIONS

The major conclusions of this paper may be summarized as follows:

There are four longitudes, located within 2-1/2 degrees of the axes fixed by the earth's elliptical equator, into which a 24 hour satellite may be placed and maintained with negligible drift for extended periods of time.

At intermediate longitudes (about 45 degrees from these "stable points") such a satellite will, even if injected "perfectly", experience a minimum of about 4 degrees of drift in 3 months following injection.

Maximum drift of a perfectly injected 24-hour satellite in an earth gravity field to 4th order differs by about 10 percent from the drift experienced in a triaxial field.

Except in the immediate vicinity of the "stable points", the initial mean daily drift in radius following a perfect injection into a 24-hour orbit is proportional to time, and the initial mean daily drift in geographic longitude is essentially proportional to the square of time.

The maximum longitude perturbational force on a 24-hour satellite in an Earth potential field to 4th order is approximately 12 percent greater than the maximum perturbation experienced in a triaxial field.

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Appendix A

List of Symbols

$A, B, C, D, E, A_1, A_2, A_3$	Perturbation constants of the uncoupled perturbation equations of motion of the near synchronous satellite of the earth
$a, b, c, d, e, f, g, h, i, j, k, l, m, n$	Perturbation constants of the coupled linearized perturbation equations of motion with respect to the synchronous circular orbit of radius r_s
d	The total differential operator
$\vec{F}$	The earth's gravity field
g_s	The acceleration of earth gravity at the near synchronous radius r_s. $g_s \approx 32.15\,(R_0/r_s)^2 \approx 32.15 \times .02288 = .7355$ ft/sec^2
i	In addition to a perturbation constant, used as an index to the coefficients of the drift Equations 57
$\dot{m}$	A point mass
n, m	When used in the earth potential function; indicates the harmonic of order n and power m
q	An index for the coefficients of the drift Equations 57. Also, a characteristic solution of the uncoupled motion Equations 36
R_0 or R_o	The mean equatorial radius of the earth ellipsoid
r, λ, ϕ	Spherical coordinates of the near synchronous satellite; geocentric radius; geographic longitude with respect to the greenwich meridian, and geocentric latitude from the earth's equator
r_1, λ_1, ϕ_1	Dimensionless perturbation coordinates

$$r_1 = \frac{\Delta r}{r_s}, \quad \lambda_1 = \Delta\lambda \text{ (radians)}, \quad \phi_1 = \Delta\phi \text{ (radians)}$$

$\Delta r, \Delta\lambda, \Delta\phi$	Perturbation coordinates in radius, geographic longitude and latitude from a reference equatorial orbit which is circular, having the near synchronous radius r_s and a period of exactly one sidereal day. These coordinates measure the drift of a near synchronous earth satellite from a point in space at a fixed r_s from the c.m. of the earth, moving along the equator at the earth's rotation rate so as to maintain a fixed geographical longitude at all times
$\hat{r}, \hat{\theta}, \hat{\phi}$	Unit Vectors for the spherical coordinate system: r, θ, ϕ
r, θ, ϕ	Spherical coordinates of geocentric radius, inertial longitude from the vernal equinox, and geocentric latitude from the earth's equator, locating the near synchronous satellite $\underset{\cdot}{m}$ in inertial space
r_s	A nominal or calculated orbit radius at injection for a near synchronous earth satellite
s	A differential operator: $$s^o(\ ) = (\ );\quad s^1(\ ) = \frac{d(\ )}{dT},\quad s^2(\ ) = \frac{d^2(\ )}{dT},\ \text{etc.};$$ and, $s_o^o(\ ) = (\ )$ at time $T = 0$, etc.
T	Dimensionless time variable: $T = wt$, where w = the earth's sidereal rotation rate, and t is real time
t	Real time, from a zero at the point of injection of the satellite into it's near synchronous orbit
$(\dot{\ }), (\ddot{\ })$	Differentiation with respect to real time t, prior to Equation 14, and differentiation with respect to dimensionless time T in and after Equation 14
V_E	The earth's gravity potential field
$\gamma_{nm}, J_{nm}, \lambda_{nm}, \theta_{nm}$	Constants of the earth's gravity potential:
ϵ	A small parameter
$(\theta_E)_o$	The right ascension of Greenwich (GHA) at $t = 0$
$\dot{\theta}_E, w$	The earth's "constant" sidereal rotation rate
λ_0 or λ_o	The initial geographic longitude of the near synchronous satellite (i.e. the geographic longitude at injection)
μ_E	The earth's gaussian gravitational constant

Appendix B

The Earth Gravity Potential and Gravitational Force Field Used

The Gravity Potential used is the exterior potential derived in Reference 4 for geocentric spherical coordinates referenced to the earth's spin axis and it's center of mass (see Figure 1). The Harmonic series is truncated after J_{44}.

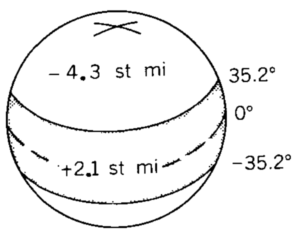
The inertial gravity constants θ_{nm} are related to the usual geographical gravity constants by;

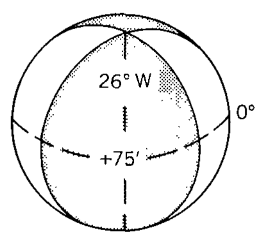
$$\theta_{nm} = (\theta_E)_0 + \dot{\theta}_E t + \lambda_{nm}$$

so that

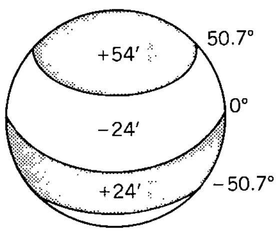
$$\theta - \theta_{nm} = \lambda - \lambda_{nm} \tag{B1}$$

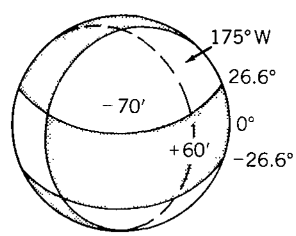
(see Figure 2). The longitudes and minimax deviations from an average earth sphere for each harmonic are taken from the potential of Kozai*. The J_{nm} and λ_{nm} of this potential are presented in Equations 47. The earth's gravity potential may be illustrated as follows:





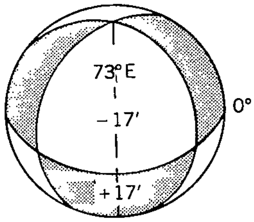
$$V_E = \frac{\mu_E}{r}\left\{1 - \frac{J_{20} R_o^2}{2r^2}\left(3\sin^2\phi - 1\right) - 3J_{22}\frac{R_o^2}{r^2}\cos^2\phi\cos 2\left(\theta - \theta_{22}\right)\right.$$

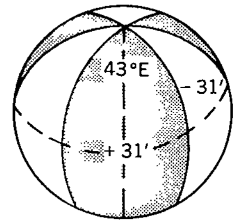




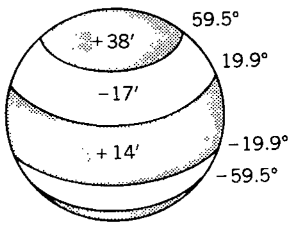
$$- \frac{J_{30} R_o^3}{2r^3}\left(5\sin^3\phi - 3\sin\phi\right) - \frac{J_{31} R_o^3}{2r^3}\cos\phi\left(15\sin^2\phi - 3\right)\cos\left(\theta - \theta_{31}\right)$$

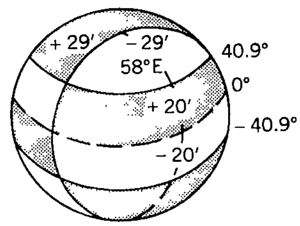
*Wagner, C. A., "The Gravitational Potential of a Triaxial Earth," Goddard Space Flight Center Document Number X-623-62-206, 1962.



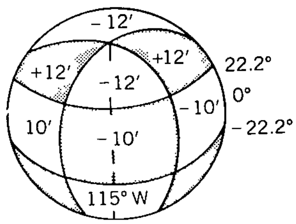


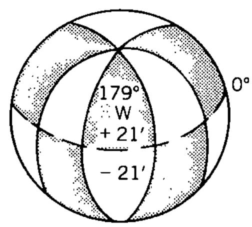
$$- 15J_{32}\frac{R_o^3}{r^3}\cos^2\phi\,\sin\phi\,\cos 2(\theta-\theta_{32}) - 15J_{33}\frac{R_o^3}{r^3}\cos^3\phi\cos 3(\theta-\theta_{33})$$



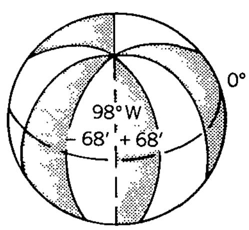


$$-\frac{J_{40}R_o^4}{8r^4}\left(35\sin^4\phi - 30\sin^2\phi + 3\right) - \frac{J_{41}R_o^4}{8r^4}\big[140\sin^3\phi - 60\sin\phi\cos\phi\cos(\theta-\theta_{41})$$





$$-\frac{J_{42}R_o^4}{8r^4}\left[420\sin^2\phi - 60\right]\cos^2\phi\cos 2(\theta-\theta_{42}) - \frac{J_{43}R_o^4}{8r^4}\left[840\sin\phi\right]\cos^3\phi\cos 3(\theta-\theta_{43})$$



$$\frac{-J_{44}R_o^4}{8r^4}\Big[840\cos^4\phi\,\cos 4(\theta-\theta_{44})\Big]\Big]\Big\}$$

$$\vec{F} = \hat{r} F_r + \hat{\theta} F_\theta + \hat{\phi} F_\phi . \tag{B2}$$

$$F_r = m G_r = m \frac{\partial V_E}{\partial r} ,$$

where

$$G_r = \frac{\mu_E}{r^2} \Bigg\{ -1 + \left(\frac{R_o}{r}\right)^2 \Bigg[\frac{3}{2} J_{20} \left(3 \sin^2 \phi - 1\right) + 9 J_{22} \cos^2 \phi \cos 2\left(\theta - \theta_{22}\right)$$

$$+ 2\left(\frac{R_o}{r}\right) J_{30} \left(5 \sin^2 \phi - 3\right)\left(\sin \phi\right) + 6\left(\frac{R_o}{r}\right) J_{31} \left(5 \sin^2 \phi - 1\right) \cos \phi \cos\left(\theta - \theta_{31}\right)$$

$$+ 60\left(\frac{R_o}{r}\right) J_{32} \cos^2 \phi \sin \phi \cos 2\left(\theta - \theta_{32}\right) + 60\left(\frac{R_o}{r}\right) J_{33} \cos^3 \phi \cos 3\left(\theta - \theta_{33}\right)$$

$$+ \frac{5}{8} \left(\frac{R_o}{r}\right)^2 J_{40} \left(35 \sin^4 \phi - 30 \sin^2 \phi + 3\right)$$

$$+ \frac{25}{2} \left(\frac{R_o}{r}\right)^2 J_{41} \left(7 \sin^2 \phi - 3\right) \cos \phi \sin \phi \cos\left(\theta - \theta_{41}\right)$$

$$+ \frac{75}{2} \left(\frac{R_o}{r}\right)^2 J_{42} \left(7 \sin^2 \phi - 1\right) \cos^2 \phi \cos 2\left(\theta - \theta_{42}\right)$$

$$+ 525\left(\frac{R_o}{r}\right)^2 J_{43} \cos^3 \phi \sin \phi \cos 3\left(\theta - \theta_{43}\right) + 525\left(\frac{R_o}{r}\right)^2 J_{44} \cos^4 \phi \cos 4\left(\theta - \theta_{44}\right) \Bigg] \Bigg\} . \tag{B3}$$

$$F_\theta = m G_\theta = \left[\frac{m}{r \cos \phi}\right] \frac{\partial V_E}{\partial \theta} ,$$

where

$$G_\theta = \frac{\mu_E}{r^2} \left(\frac{R_o}{r}\right)^2 \Bigg\{ 6 J_{22} \cos \phi \sin 2\left(\theta - \theta_{22}\right) + \frac{3}{2} \left(\frac{R_o}{r}\right) J_{31} \left[5 \sin^2 \phi - 1\right] \sin \left(\theta - \theta_{31}\right)$$

$$+ 30\left(\frac{R_o}{r}\right) J_{32} \cos \phi \sin \phi \sin 2\left(\theta - \theta_{32}\right) + 45\left(\frac{R_o}{r}\right) J_{33} \cos^2 \phi \sin 3\left(\theta - \theta_{33}\right)$$

$$+ \frac{5}{2} \left(\frac{R_o}{r}\right)^2 J_{41} \left(7 \sin^2 \phi - 3\right) \sin \phi \sin\left(\theta - \theta_{41}\right) + 15\left(\frac{R_o}{r}\right)^2 \left(7 \sin^2 \phi - 1\right)$$

$$\cdot J_{42} \cos \phi \sin 2\left(\theta - \theta_{42}\right) + 315\left(\frac{R_o}{r}\right)^2 J_{43} \cos^2 \phi \sin \phi \sin 3\left(\theta - \theta_{43}\right)$$

$$+ 420\left(\frac{R_o}{r}\right)^2 J_{44} \cos^3 \phi \sin 4\left(\theta - \theta_{44}\right) \Bigg\} . \tag{B4}$$

$$F_\phi = \underset{\cdot}{m} G_\phi = \frac{\underset{\cdot}{m}}{r} \frac{\partial V_E}{\partial \phi},$$

where,

$$\begin{aligned} G_\phi = \frac{\mu_E}{r^2}\left(\frac{R_o}{r}\right)^2 \Bigg\{ & -3J_{20} \sin\phi \cos\phi + 6J_{22} \cos\phi \sin\phi \cos 2\left(\theta - \theta_{22}\right) \\ & - \frac{3}{2}\left(\frac{R_o}{r}\right) J_{30} \left(5\sin^2\phi - 1\right) \cos\phi + \frac{3}{2}\left(\frac{R_o}{r}\right) J_{31} \left(15\sin^2\phi - 11\right) \sin\phi \cos\left(\theta - \theta_{31}\right) \\ & + 15\left(\frac{R_o}{r}\right) J_{32} \left(3\sin^2\phi - 1\right) \cos\phi \cos 2\left(\theta - \theta_{32}\right) \\ & + 45\left(\frac{R_o}{r}\right) J_{33} \cos^2\phi \sin\phi \cos 3\left(\theta - \theta_{33}\right) - \frac{5}{2}\left(\frac{R_o}{r}\right)^2 J_{40} \left(7\sin^2\phi - 3\right) \\ & \cdot \sin\phi \cos\phi + \frac{5}{2}\left(\frac{R_o}{r}\right)^2 J_{41} \left(28\sin^4\phi - 27\sin^2\phi + 3\right) \cos\left(\theta - \theta_{41}\right) \\ & + 30\left(\frac{R_o}{r}\right)^2 J_{42} \left(7\sin^2\phi - 4\right) \cos\phi \sin\phi \cos 2\left(\theta - \theta_{42}\right) \\ & + 105\left(\frac{R_o}{r}\right)^2 J_{43} \left(4\sin^2\phi - 1\right) \cos^2\phi \cos 3\left(\theta - \theta_{43}\right) \\ & + 420\left(\frac{R_o}{r}\right)^2 J_{44} \cos^3\phi \sin\phi \cos 4\left(\theta - \theta_{44}\right) \Bigg\}. \end{aligned} \tag{B5}$$

The radial perturbation of the gravitational field referred to in this report is the residual of the sum of the gravitational and centrifugal forces on $\underset{\cdot}{m}$ at the moment of injection at the radius r_s. Since $\underset{\cdot}{m}$ is injected at an angular rate w for conditions where the initial perturbation rates with respect to the reference synchronous orbit are zero, the centrifugal force on $\underset{\cdot}{m}$ at the moment of this injection at r_s is $+\hat{r}\left(w^2 r_s \underset{\cdot}{m}\right)$. Thus, the radial perturbation force, as defined above, at the moment of injection is: radial perturbation force $= \underset{\cdot}{m} w^2 r_s + \underset{\cdot}{m} G_{r_s} = \underset{\cdot}{m} w^2 r_s e$. Thus, the condition for the radial perturbation force (for a "perfect" injection, at injection) to be zero is: $e = 0$.

Appendix C

Magnitudes of the Longitude and Latitude Perturbation Forces per Unit Mass on a Near Synchronous Equatorial Satellite due to Gravity

It may be verified from Appendix B, that:

1. The longitude perturbation force per unit mass on the near synchronous satellite is given by

$$\left(G_\theta\right)_{r_s} \doteq \left(g_s\right)\left(a\right) . \tag{C1}$$

2. The latitude perturbation force per unit mass on the near synchronous satellite is given by

$$\left(G_\phi\right)_{r_s} \doteq \left(g_s\right)\left(i\right) , \tag{C2}$$

where g_s is the radial acceleration of gravity in a $1/r^2$ earth field at the synchronous radius:

$$g_s = \frac{\mu_E}{r_s^2} = \frac{\mu_E}{\left(R_o\right)^2}\left(\frac{R_o}{r_s}\right)^2 = 32.15 \times .02288 = .735 \text{ ft./sec.}^2 \tag{C3}$$

It is noted that at the synchronous radius

$$\frac{\mu_E}{w^2 r_s^3} \approx 1 . \tag{C4}$$

In Figure C1, the perturbation forces per unit mass are plotted for two earth gravity fields. One is for a field through 4th order due to Kozai*, and the other is for a simpler triaxial field which includes only the second order harmonic potential constants from the same source. It is seen that, while the triaxial components clearly dominate the longitude perturbation force function, the maximum force in the "full" field is 12 percent greater than the maximum force in the triaxial (Table C1). The latitude perturbation in the triaxial field is, of course, zero. It is noted that the extreme magnitudes of the latitude perturbation function around the equator are about one order of magnitude less than the

*Kozai, Y., Private Communication, November 1962.

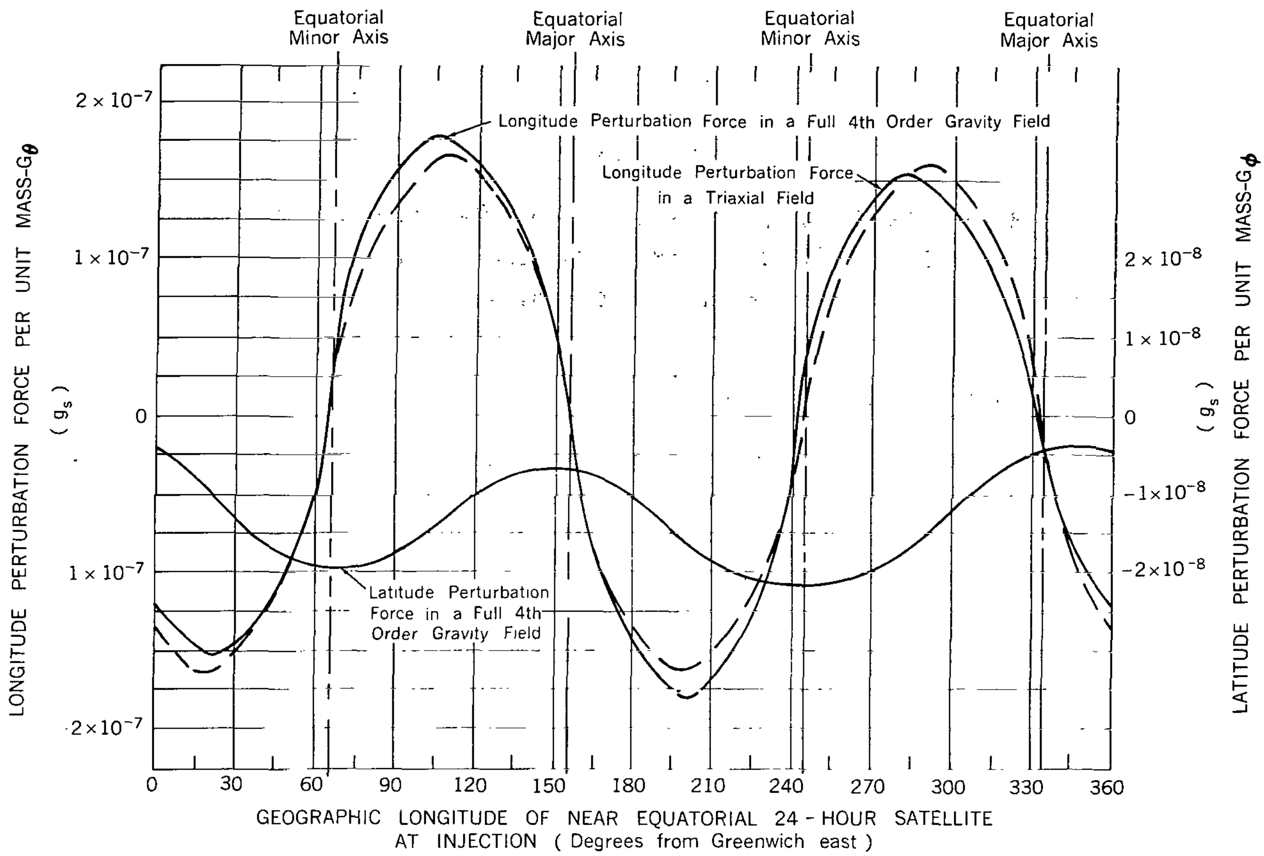


Figure C1—The longitude and latitude perturbation forces on a near Equatorial 24-hour satellite around the equator, due to the geopotential of Kozai.

Table C1

Comparison of Maximum Longitudinal Perturbation Forces at Near Synchronous Altitudes Between the Triaxial and Full 4th Order Gravity Fields of Kozai.

Field	Minimax $(G_\theta)_{r_s}$ (units of g_s)	λ_o (degrees)	Percent difference of minimax G_θ between full and triaxial field (full-triax/triax) x 100
Triaxial Full	-1.65×10^{-7} -1.51×10^{-7}	18.6 21.7	+8.5
Triaxial Full	$+1.65 \times 10^{-7}$ $+1.78 \times 10^{-7}$	108.6 109.7	+7.9
Triaxial Full	-1.65×10^{-7} -1.85×10^{-7}	198.9 200.0	-12.0
Triaxial Full	$+1.65 \times 10^{-7}$ $+1.53 \times 10^{-7}$	288.9 282.3	-7.3

extremes of the longitude perturbation function. This fact gives justification to the assumption that the regions of stability for the near synchronous satellite may be considered to be fixed by the longitude perturbation alone (i.e., by the zeros of n). It is also interesting that the minimum of the latitude perturbation in the full field at synchronous radius occurs close to the dynamically unstable region near the earth equator's major axis while the maximum latitude perturbation occurs near the dynamically stable region close to the minor axis. This coincidence tends to minimize the drift in the neighborhood of the major axis.

Appendix D

Procedure for Determining the Injection Radius and Longitude into a Near Synchronous Orbit with Minimal Initial Perturbation Accelerations

The two conditions for a minimal drift, near synchronous earth orbit, are assumed to be

$$e = 0 , \tag{D1}$$

$$n = 0 . \tag{D2}$$

These two conditions give a set of four injection radii and longitudes for minimum drift in a near synchronous equatorial earth orbit, with the potential of Kozai.* It can be assumed that injection at the earth rate into an inclined orbit whose nodes are near these geographic longitudes at near synchronous radii, will give orbits with similarly small nodal drift if the inclination is not excessive.

Let;

$$r_s = r_o + \Delta r_s , \tag{D3}$$

where $r_s << r_o$ is assumed and where r_o is the solution to:

$$\frac{w^2 r_0^3}{\mu_E} = 1 + \frac{3}{2} J_{20} \left(\frac{R_o}{r_o}\right)^2 - \frac{15}{8} J_{40} \left(\frac{R_o}{r_o}\right)^4 \tag{D4}$$

r_o may be thought of as the synchronous radius for the "oblate" earth. With Equations D3 and D4 into Equations D1 and D2, by ignoring all powers of $\Delta r_s/r_o$ greater than one and all terms in $J_{nm} \Delta r_s/r_o$; the synchronous conditions become

$$\begin{aligned}\frac{\Delta r_s}{r_o} = \Big\{ & (K_{31})_1 \cos\lambda_o + (K_{31})_2 \sin\lambda_o + \left[(K_{22})_1 + (K_{42})_1\right] \cos 2\lambda_o + \Big[(K_{22})_2 \\ & + (K_{42})_2\Big] \sin 2\lambda_o + (K_{33})_1 \cos 3\lambda_o + (K_{33})_2 \sin 3\lambda_o + (K_{44})_1 \cos 4\lambda_o \\ & + (K_{44})_2 \sin 4\lambda_o \Big\} \Big/ \frac{3w^2 r_o^3}{\mu_E} ,\end{aligned} \tag{D5}$$

*Kozai, Y., Private Communication, November 1962.

and

$$0 = (K_{31})_3 \sin\lambda_o + \left[(K_{22})_3 + (K_{42})_3\right] \sin 2\lambda_o + (K_{33})_3 \sin 3\lambda_o + (K_{44})_3 \sin 4\lambda_o$$

$$+ (K_{31})_4 \cos\lambda_o + \left[(K_{22})_4 + (K_{42})_4\right] \cos 2\lambda_o + (K_{33})_4 \cos 3\lambda_o$$

$$+ (K_{44})_4 \sin 4\lambda_o \ . \qquad \text{(D6)}$$

In Equations D5 and D6, the gravitational constants $(K_{nm})_i$ are

$$(K_{31})_1 = -6J_{31}\left(\frac{R_o}{r_o}\right)^3 \cos\lambda_{31} \ ,$$

$$(K_{22})_1 = -9J_{22}\left(\frac{R_o}{r_o}\right)^2 \cos 2\lambda_{22} \ ,$$

$$(K_{42})_1 = \frac{75}{2}J_{42}\left(\frac{R_o}{r_o}\right)^4 \cos 2\lambda_{42} \ ,$$

$$(K_{33})_1 = -60J_{33}\left(\frac{R_o}{r_o}\right)^3 \cos 3\lambda_{33} \ ,$$

$$(K_{44})_1 = -525J_{44}\left(\frac{R_o}{r_o}\right)^4 \cos 4\lambda_{44} \ , \qquad \text{(D7)}$$

$$(K_{31})_3 = -\frac{3}{2}J_{31}\left(\frac{R_o}{r_o}\right) \cos\lambda_{31} \ ,$$

$$(K_{22})_3 = 6J_{22} \cos 2\lambda_{22} \ ,$$

$$(K_{42})_3 = -15J_{42}\left(\frac{R_o}{r_o}\right)^2 \cos 2\lambda_{42} \ ,$$

$$(K_{33})_3 = 45J_{33}\left(\frac{R_o}{r_o}\right) \cos 3\lambda_{33} \ ,$$

$$(K_{44})_3 = 420J_{44}\left(\frac{R_o}{r_o}\right)^2 \cos 4\lambda_{44} \ ,$$

where

$$(K_{nm})_2 = (K_{nm})_1 \operatorname{Tan}(n\lambda_{nm}) \ , \text{ and; } (K_{nm})_4 = -(K_{nm})_3 \operatorname{Tan}(n\lambda_{nm}) \ .$$

To solve Equation D4 for the oblate earth synchronous radius r_o, let

$$r_o = r_{01} + \Delta r_o \, , \tag{D8}$$

where r_{01} is the "spherical earth" synchronous radius satisfying the expression

$$\frac{w^2 r_{01}^3}{\mu_E} = 1 \, . \tag{D9}$$

If $\Delta r_o << r_{01}$ is assumed, then with Equations D9 and D8 into Equation D4, by ignoring the $J_{nm} \Delta r_o / r_{01}$ as of second order, the oblate synchronous radius is determined from

$$\frac{\Delta r_o}{r_{01}} = \left(\frac{R_o}{r_{01}}\right)^2 \left[\frac{3}{2} J_{20} - \frac{15}{8} J_{40} \left(\frac{R_o}{r_{01}}\right)^2\right] / 3 \, . \tag{D10}$$

The procedure to determine the elements of the "stable" synchronous orbit is:

1. Solve for the spherical synchronous radius r_{01} from Equation D9.
2. Solve for the oblate synchronous radius r_o from the results of the solution for Δr_o from Equation D10 into Equation D8.
3. Solve for the constants K in Equation D7.
4. Solve for the "stable" synchronous injection or nodal longitudes λ_o from Equation D6.
5. Solve for the "stable" synchronous radii r_s corresponding to these longitudes from the results of the solution for Δr_s from Equation D5 in Equation D3.

For the earth potential used, the magnitude assumptions in Equations D1 and D2 are valid and the aforementioned uncoupled procedure establishes the "stable" synchronous elements to high accuracy.